International Journal of Theoretical Physics, Vol. 39, No. 3, 2000

Effect Algebras and Para-Boolean Manifolds†

G. Cattaneo,[1] M. L. Dalla Chiara,[2] R. Giuntini,[3] and S. Pulmannova[4]

Received December 8, 1999

It is shown that every effect algebra can be represented as a pasting of a system where each element is the range of an unsharp observable. To describe the range of an unsharp observable algebraically, the notion of a "para-Boolean quasi-effect algebra" is introduced. Some intrinsic compatibility conditions ensuring commensurability of effects are studied.

1. THE ALGEBRAIC STRUCTURE OF THE RANGE OF AN UNSHARP OBSERVABLE

It is well known that every orthomodular poset can be represented as the pasting of a system of Boolean algebras [5, 21, 17], where the range of every observable is a Boolean algebra [17] (in particular, each *separable* element of the system of Boolean algebras is the range of a (real) sharp observable [22]). The aim of the paper is to generalize such a result to the case of effect algebras and unsharp observables. In contrast to the sharp case, the range of an unsharp observable not only fails to be a Boolean algebra, but is not even closed under the orthogonal sum. However, the range of any unsharp observable retains some classical features that are captured by the abstract notion of *para-Boolean quasi-effect algebra*. One can prove that every effect algebra can be represented as a pasting of a system where each

† This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

[1] Dipartimento di Scienze dell'Informazione, Università di Milano, I-20135 Milan, Italy; e-mail: cattang@dsi.unimi.it.
[2] Dipartimento di Filosofia, Università di Firenze, I-50139, Florence, Italy; e-mail: dachiara@mailserver.idg.fi.cnr.it.
[3] Dipartimento di Scienze Pedagogiche e Filosofiche, Università di Cagliari, 1-09123 Cagliari, Italy; e-mail: giuntini@philos.unifi.it.
[4] Mathematical Institute, Slovak Academy of Sciences, SK-81473, Bratislava, Slovakia; e-mail: pulmann@mau.savba.sk.

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element of the system is the range of an unsharp observable. From the physical point of view, effects contained in the range of an observable are *coexistent*, in the sense they are *simultaneously measurable*.

Definition 1.1. An *effect algebra* is a partial structure $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0}\rangle$ where $\oplus$ is a partial binary operation on A. When $\oplus$ is defined for a pair a, $b \in A$, we will write $\exists(a \oplus b)$. The following conditions hold:

- (i) *Commutativity*:
  $\exists(a \oplus b)$ implies $\exists(b \oplus a)$ and $a \oplus b = b \oplus a$.
- (ii) *Associativity*:
  $[\exists(b \oplus c)$ and $\exists(a \oplus (b \oplus c))]$ implies $[\exists(a \oplus b)$ and $\exists((a \oplus b) \oplus c)$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c]$.
- (iii) *Strong excluded middle*:
  For any a, there exists a unique x such that $a \oplus x = \mathbf{1}$.
- (iv) *Consistency*:
  $\exists(a \oplus \mathbf{1})$ implies $a = \mathbf{0}$.

An orthogonality relation $\perp$, a partial order relation $\leq$, and a generalized complement $'$ can be defined in any effect algebra.

Definition 1.2. Let $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0}\rangle$ be an effect algebra and let $a, b \in A$.

- (i) $a \perp b$ iff $a \oplus b$ is defined in A.
- (ii) $a \leq b$ iff $\exists c \in A$ such that $a \perp c$ and $b = a \oplus c$.
- (iii) The *generalized complement* of a is the unique element a' such that $a \oplus a' = \mathbf{1}$ (the definition is justified by the strong excluded middle condition).

The structure $\langle A, \leq, ', \mathbf{1}, \mathbf{0}\rangle$ is an involutive bounded poset (also de Morgan poset).

The category of all effect algebras turns out to be (categorically) equivalent to the category of all *difference posets* [16], which were first studied in ref. 10.

Let $\mathcal{H}$ be a Hilbert space, and denote by $E(\mathcal{H})$ the set of all self-adjoint operators E s.t. $\mathbf{0} \leq E \leq \mathbf{1}$ (where $\mathbf{0}$ and $\mathbf{1}$ represent the null and the identity operator, respectively). In order to induce the structure of an effect algebra on $E(\mathcal{H})$, it is sufficient to define a partial sum $\oplus$ as follows:

$$\exists(E \oplus F) \quad \text{iff } E + F \in E(\mathcal{H}), \qquad \text{and in this case} \quad E \oplus F := E + F$$

where $+$ is the usual sum-operator.

It turns out that the structure $\langle E(\mathcal{H}), \oplus, \mathbf{1}, \mathbf{0}\rangle$ is an effect algebra, where the generalized complement of any effect E is just $\mathbf{1} - E$.

Definition 1.3. Let $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0}\rangle$ be an effect algebra. Let $(\Omega, \mathcal{B}(\Omega))$ be a pair consisting of a nonempty set Ω and a Boolean algebra $\mathcal{B}(\Omega)$ of subsets of Ω. A $(\mathcal{B}(\Omega), \mathcal{A})$-*observable* is a map $\alpha: \mathcal{B}(\Omega) \to A$ satisfying the following conditions:

1. $\alpha(\Omega) = 1$.
2. $\Delta_1 \cap \Delta_2 = \emptyset$ implies $\alpha(\Delta_1 \cup \Delta_2) = \alpha(\Delta_1) \oplus \alpha(\Delta_2)$.

If no confusion is possible, by "observable" we will mean a $(\mathcal{B}(\Omega), \mathcal{A})$-observable, where both the Boolean algebra of subsets and the effect algebra are fixed. When ω is an element of Ω s.t. $\{\omega\} \in \mathcal{B}(\Omega)$, we will write, for the sake of simplicity, $\alpha(\omega)$ instead of $\alpha(\{\omega\})$. When $\Omega = \mathbb{R}$ and $\mathcal{B}(\mathbb{R})$ is the standard Borel σ-*algebra* of subsets of $\mathbb{R}$, we will speak of *real observables*. Let α be an observable and let $Range(\alpha)$ denote its range. In other words,

$$Range(\alpha) = \{a \in \mathcal{A} | \exists \Delta \in \mathcal{B}(\Omega): a = \alpha(\Delta)\}$$

Lemma 1.1. There exists an observable α s.t.:

1. $Range(\alpha)$ is not closed under $\oplus$.
2. $Range(\alpha)$ does not satisfy the associativity condition [(ii) of Definition 1.1].
3. Let $a, b \in Range(\alpha)$. The relation $\leq$ [where $a \leq b$ iff $\exists c \in Range(\alpha)$ s.t. $\exists(a \oplus c)$ and $b = a \oplus c$] does not define a partial order in $Range(\alpha)$.

Proof. Let $\mathcal{A}$ be the unit-interval effect algebra. Let $\Omega := \{1, 2, 3, 4\}$ and let $\mathcal{B}(\Omega)$ be the power-set of Ω. Let α be the observable defined as follows:

$$\alpha(1) = \frac{3}{12}, \qquad \alpha(2) = \frac{4}{12}$$

$$\alpha(3) = \frac{1}{12}, \qquad \alpha(4) = \frac{4}{12}$$

It turns out that

$$Range(\alpha) = \left\{0, 1, \frac{3}{12}, \frac{4}{12}, \frac{1}{12}, \frac{7}{12}, \frac{5}{12}, \frac{8}{12}, \frac{9}{12}, \frac{11}{12}\right\}$$

1. $Range(\alpha)$ is not closed under $\oplus$. For example, $\alpha(1) \oplus \alpha(1) = \frac{6}{12} \notin Range(\alpha)$.

2. Associativity fails. For example, $\exists(\alpha(1) \oplus \alpha(3))$, $\exists(\alpha(1) \oplus (\alpha(1) \oplus \alpha(3)))$, and $\exists(\alpha(1) \oplus (\alpha(1) \oplus \alpha(3))) = \frac{7}{12}$; however, $\alpha(1) \oplus \alpha(1) = \frac{6}{12} \notin Range(\alpha)$.

3. The relation $\lessdot$ is not transitive. For example, $\frac{1}{12} \lessdot \frac{4}{12}$ since $\frac{1}{12} \oplus \frac{3}{12} = \frac{4}{12}$ and $\frac{3}{12} \in Range(\alpha)$. Further, $\frac{4}{12} \lessdot \frac{7}{12}$ since $\frac{4}{12} \oplus \frac{3}{12} = \frac{7}{12}$. But $\frac{1}{12} \not\lessdot \frac{7}{12}$ since $\frac{1}{12} \oplus \frac{6}{12} = \frac{7}{12}$ and $\frac{6}{12} \notin Range(\alpha)$. ■

As a consequence, $Range(\alpha)$ is not a subeffect algebra of $\mathcal{A}$.

Definition 1.4. A *quasi effect algebra* is a structure

$$\mathcal{A} = \langle A, \oplus, \leq, ', \mathbf{1}, \mathbf{0} \rangle$$

where:

1. $\langle A, \leq, ', \mathbf{1}, \mathbf{0} \rangle$ is an involutive bounded poset.
2. $\exists(a \oplus b)$ implies $\exists(b \oplus a)$ and

$$a \oplus b = b \oplus a$$

3. $a \oplus a' = \mathbf{1}$.
4. $\exists(a \oplus \mathbf{1})$ implies $a = \mathbf{0}$.
5. $a \oplus \mathbf{0} = a$.
6. $a \leq b$ and $\exists(a \oplus c)$ and $\exists(b \oplus c)$ imply $a \oplus c \leq b \oplus c$.

Lemma 1.2. Let $\mathcal{A}$ be an effect algebra, where $\leq$ and $'$ represent the induced partial order relation and the fuzzy complement, respectively. Let α be an $\mathcal{A}$-valued observable. The range of α gives rise to a quasi effect algebra:

$$\langle Range(\alpha), \oplus_\alpha, \leq_\alpha, '^\alpha, \mathbf{1}_\alpha, \mathbf{0}_\alpha \rangle$$

where:

1. $\exists(a \oplus_\alpha b)$ iff $\exists(a \oplus b)$ and $(a \oplus b) \in Range(\alpha)$. If $\exists(a \oplus_\alpha b)$, then

$$a \oplus_\alpha b := a \oplus b$$

2. $\leq_\alpha$ is the restriction of $\leq$ to $Range(\alpha)$.
3. $'^\alpha = '$ (recall that the $Range(\alpha)$ is closed under $'$).
4. $\mathbf{1}_\alpha = \mathbf{1}$, $\mathbf{0}_\alpha = \mathbf{0}$.

Notice that it may happen that $a \leq_\alpha b$ even if there exists no $c \in Range(\alpha)$ such that $a \oplus_\alpha c = b$.

Lemma 1.3 [12]. There exists an observable α s.t. $Range(\alpha)$ is not a lattice with respect to the partial order $\leq_\alpha$ defined in Lemma 1.2.

Definition 1.5. A quasi effect algebra

$$\mathcal{A} = \langle A, \oplus_A, \leq_A, {}'^A, \mathbf{1}_A, \mathbf{0}_A \rangle$$

is *para-Boolean* iff there exists an effect algebra $\mathcal{B} = \langle B, \oplus_B, {}'^B, \mathbf{1}_B, \mathbf{0}_B \rangle$ such that:

1. $A \subseteq B$.
2. $a, b \in A$ and $\exists(\alpha \oplus_A b)$ imply $\exists(a \oplus_B b)$ and $a \oplus_A b = a \oplus_B b$.
3. $a \in A$ implies $a'^B = a'^B$
4. $a, b \in A$ implies $[a \leq_A b$ iff $a \leq_B, b]$.
5. $\mathbf{0}_A = \mathbf{0}_B$, $\mathbf{1}_A = \mathbf{1}_B$.
6. There exists a $\mathcal{B}$-observable α such that $A \subseteq Range(\alpha)$.

In other words, a para-Boolean quasi effect algebra is contained in the range of an observable.

Definition 1.6. A *quasi effect manifold* is a system of quasi effect algebras

$$\{\mathcal{A}_i = \langle A_i, \oplus_i, \leq_i, {}'^i, \mathbf{1}_i, \mathbf{0}_i \rangle : i \in I\}$$

such that:

1. $\forall ij\ [\mathbf{1}_i = \mathbf{1}_j$ and $\mathbf{0}_i = \mathbf{0}_j]$.
2. $a, b \in A_i \cap A_j$ and $\exists(a \oplus_i b)$ and $\exists(a \oplus_j b)$ imply $a \oplus_i b = a \oplus_j b$.
3. $a \in A_i \cap A_j$ implies $a'^i = a'^j$.
4. $\exists ji\ [b, c, b \oplus_j c \in A_j$ and $a, b \oplus_j c, a \oplus_i (b \oplus_j c) \in A_i]$ implies $\exists hk\ [a, b, a \oplus_h b \in A_h$ and $a \oplus_h b, c, (a \oplus_h b) \oplus_k c \in A_k]$ and

$$a \oplus_i (b \oplus_j c) = (a \oplus_h b) \oplus_k c$$

5. $a, b \in A_i \cap A_j$ implies $[a \leq_i b$ iff $a \leq_j b]$.

Definition 1.7. A *para-Boolean* manifold is a quasi effect manifold, where each quasi effect algebra is para-Boolean.

Theorem 1.1. Every quasi effect manifold $\{\mathcal{A}_i\}$ determines an effect algebra $\mathcal{B} = \langle B, \oplus, \mathbf{0}, \mathbf{1} \rangle$ s.t.:

1. $B = \cup_i A_i$
2. $\exists(a \oplus b)$ iff $\exists i\ a, b \in A_i$ and $\exists(a \oplus_i b)]$. If $\exists(a \oplus b)$ and $\exists(a \oplus_i b)$, then $a \oplus b = a \oplus_i b_i$.
3. $\mathbf{1} = \mathbf{1}_i$, $\mathbf{0} = \mathbf{0}_i$.

We say that $\{A_i\}$ is a covering of $\mathcal{B}$. ■

Proof. Straightforward.

Theorem 1.2. Every effect algebra is covered by a para-Boolean manifold.

Proof. Let $\mathcal{B} = \langle B, \oplus, \mathbf{1}, \mathbf{0}\rangle$ be an effect algebra. Let us consider the class of all real $\mathcal{B}$-observables. Each $Range(\alpha)$ determines a para-Boolean quasi-effect algebra

$$\mathcal{A}_i = \langle Range(\alpha_i), \oplus, \leq, ', \mathbf{1}, \mathbf{0}\rangle$$

where $\oplus \leq, ', \mathbf{1}, \mathbf{0}$ are the operations (relations) of $\mathcal{B}$ restricted to $Range(\alpha_i)$. Let us consider $\{\mathcal{A}_i\}$. We will prove that $\{\mathcal{A}_i\}$ is a para-Boolean manifold which covers $\mathcal{B}$.

Conditions 1–3 and 5 of Definition 1.6 are easily verified.

Now, we prove condition 4. Suppose $\exists(b \oplus_j, c)$ and $\exists(a \oplus_i (b \oplus_j c))$. Then $\exists(b \oplus c)$ and $\exists(a \oplus (b \oplus c))$. Thus, $b \perp_B c$ and $a \perp_B (b \oplus c)$. Then, by associativity of $\mathcal{B}$, $\exists(a \oplus_B b)$ and $\exists((a \oplus_B b) \oplus_B c)$. Since $a \perp b$, there must be an observable α_h s.t. $a, b, a \oplus b \in Range(\alpha_h)$. Similarly, since $a \oplus_h b \perp c$, there must be an observable α_k s.t. $a \oplus_h b, c \in Range(\alpha_k)$ and $(a \oplus_h b) \oplus_k c \in Range(\alpha_k)$. Then, clearly, $a \oplus_i (b \oplus_j c) = (a \oplus_h b) \oplus_k c$. Consequently, $\{\mathcal{A}_i\}$ is a para-Boolean manifold. We now prove that $\{\mathcal{A}_i\}$ is a covering of $\mathcal{B}$. Clearly, $B = \cup_i Range(\alpha_i)$.

If $\exists(a \oplus_B b)$, then $a \perp_B b$, so that there exists an observable α_i s.t. a, b, $a \oplus_i b \in Range\ (\alpha_i)$ and $a \oplus_i b = a \oplus_B b$. ∎

2. DIFFERENT NOTIONS OF COMPATIBILITY

From the physical point of view a privileged notion of compatibility is represented by the relation of *commeasurability* (called also *simultaneous measurability* or *coexistence*). Consider an effect algebra $\mathcal{A}$. Let $\mathrm{Obs}^{\mathcal{A}}$ denote the set of all observables on $\mathcal{A}$ and let $a, b \in$ A. We will follow Varadarajan [22, 23, p. 118] (see also Mackey [15, p. 70]).

Definition 2.1. a and b are called *commeasurable* (a ♠ b iff $\exists \mathcal{O} \in Obs^{\mathcal{A}}$: $a, b \in Range(\mathcal{O})$.

It would be desirable to have an *intrinsic* definition of commeasurability. As is well known, in orthomodular lattices, such a condition is represented by a decomposition property [22, 15]. In the effect algebra case one can define this intrinsic notion as follows [9]:

Definition 2.2. a and b are called *Mackey-compatible* (a $♠_M$ b) iff $\exists a_1$, b_1, c [$\exists(a_1 \oplus b_1 \oplus c)$ and $a = a_1 \oplus c$ and $b = b_1 \oplus c$]. In such a case, we will say that a_1, b_1, c represent a *Mackey decomposition* of a, b.

One can easily show the following result:

Lemma 2.1. a ♠ b iff a $♠_M$ b.

By transferring the usual notion of compatibility from orthomodular posets to effect algebras, we obtain the following weaker relation.

Definition 2.3. a and b are called *weakly-Mackey compatible* ($a \spadesuit_{WM} b$) iff $\exists a_1 b_1 c\ [a_1 \perp b_1,\ a_1 \perp c,\ b_1 \perp c$, and $a = a_1 \oplus c$, and $b = b_1 \oplus c]$. In such a case, we will say that a_1, b_1, c represent a *Mackey weak decomposition* of a, b.

Lemma 2.2. Mackey-compatibility implies weakly-Mackey compatibility, but not the other way round.

The following example shows that weakly-Mackey compatibility is not sufficient for Mackey-compatibility.

Example 2.1. Recall that a *Wright triangle* is an orthoalgebra which is a pasting of three blocks determined by the following sets of atoms: $\{a, b, c\}$, $\{c, d, e\}$, $\{e, f, a\}$. We have $a \perp c$, $c \perp e$, $e \perp a$, $b' = a \oplus c$, $d' = c \oplus e$. At the same time, b', d' are not Mackey-compatible since they do not belong to one and the same block. Therefore, the *Wright triangle* represents an example of an effect algebra where the relation of Mackey-compatibility and weakly-Mackey compatibility do not coincide.

Moreover, we have the following result:

Lemma 2.3. $\mathcal{A}$ is an orthomodular poset if and only if the following condition is satisfied:

- $\forall a, b \in \mathcal{A}$, $a = a_1 \oplus c$, $b = b_1 \oplus c$ is a Mackey weak decomposition iff it is a Mackey decomposition of a, b.

Proof. It is easy to see that an effect algebra $\mathcal{A}$ is an orthomodular poset iff for any $a, b, c \in \mathcal{A}$, $a \perp b$, $b \perp c$, and $c \perp a$ imply $a \oplus b \perp c$. Indeed, the latter property implies that $a \oplus b$ coincides with $a \vee b$ whenever $a \perp b$.

Clearly, if $\mathcal{A}$ is an orthomodular poset, then every Mackey weak decomposition is a Mackey decomposition.

Conversely, if $\mathcal{A}$ is not an orthomodular poset, then there are pairwise orthogonal elements a, b, c such that $a \oplus b$ is not orthogonal to c. Consider $u = a \oplus b$, $v = a \oplus c$; then a, b, c is a Mackey weak decomposition, but not a Mackey decomposition, of u, v. ■

In any orthomodular poset, Mackey decompositions are unique (whenever they exist). In effect algebras, instead, Mackey decompositions need not be unique.

Example 2.2. Consider the chain $\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ and the elements $\frac{1}{4}, \frac{3}{4}$.

Then both $0, \frac{1}{4}, \frac{3}{4}$ and $\frac{1}{4}, 0, \frac{1}{2}$ are Mackey decompositions of $\frac{1}{4}$ and $\frac{3}{4}$. For a counterexample in orthoalgebras see the *Frazer cube* [14].

Let us now introduce some properties that concern the whole effect algebra $\mathscr{A}$.

Definition 2.4. An effect algebra $\mathscr{A}$ satisfies the *Mackey property* iff the relation $\spadesuit_M$ is universal.

Definition 2.5. An effect algebra $\mathscr{A}$ satisfies the *Riesz property* iff $\forall xyz$: $x \leq y \oplus z$ implies $\exists y_1 z_1$ $[y_1 \leq y$ and $z_1 \leq z$ and $x = y_1 \oplus z_1]$.

Lemma 2.4 [20]. If an effect algebra $\mathscr{A}$ satisfies the Riesz property, then $\mathscr{A}$ is an interval-effect algebra, but not the other way round.

As expected, the concrete effect algebra $E(\mathscr{H})$ does not satisfy either the Mackey property or the Riesz property.

Lemma 2.5. If an effect algebra A satisfies the Riesz property, then A satisfies the Mackey property, but not the other way round.

Proof. For any a, b we have $a \leq b \oplus b'$. By the Riesz property, there are $a_1 \leq b$, $a_2 \leq b'$ such that $a = a_1 \oplus a_2$. Since $a_1 \leq b$, there is c such that $b = a_1 \oplus c$. Now $a_1 \oplus c \leq b$, $a_2 \leq b'$ imply $a_1 \oplus c \perp a_2$, so that a_1, a_2, c is a Mackey decomposition of a, b.

The *Fano plane* (Fig. 1) represents an example of an effect algebra where the Mackey property is satisfied, whereas the Riesz property fails. ■

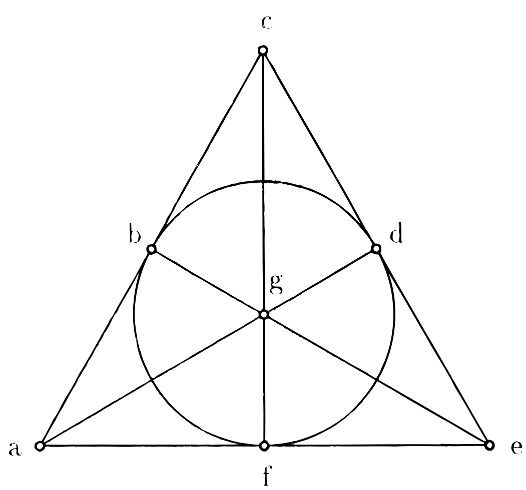

Fig. 1. Fano plane.

The following definition provides a strengthening of the Mackey compatibility which is equivalent to the Riesz property.

Definition 2.6. Let x be any element of $\mathcal{A}$. Two elements a, b are called *x-Mackey compatible* ($a \spadesuit_x b$) iff $a, b \leq x$ and there is a Mackey decomposition a_1, b_1, c of a, b such that $a_1 \oplus b_1 \oplus c \leq x$.

Theorem 2.1. An effect algebra $\mathcal{A}$ satisfies the Riesz property iff for any $a, b, x, \in \mathcal{A}$, $a, b \leq x$ implies $a \spadesuit_x b$.

Proof. Assume the Riesz property. Let $a, b \leq x$. Then there is y such that $a \leq b \oplus y = x$. By hypothesis, there are $c \leq b$, $a_1 \leq y$ such that $a = c \oplus a_1$. Let b_1 be such that $c \oplus b_1 = b$. Now $c \oplus b_1 = b$, $a_1 \leq y$ imply $a_1 \oplus b_1 \oplus c \leq b \oplus y = x$. Hence $a \spadesuit_x b$.

To prove the converse, let $x \leq y \oplus z$. Put $w := y \oplus z$. By hypothesis, $x \spadesuit_w y$, hence there are x_1, y_1, u such that $x_1 \oplus y_1 \oplus u \leq w$ and $x = x_1 \oplus u$, $y = y_1 \oplus u$. It suffices to prove $x_1 \leq z$. Let v be such that $x_1 \oplus y_1 \oplus u \oplus v = w = y \oplus z$. By the cancellation property, $x_1 \oplus v = z$, hence $x_1 \leq z$. ■

Theorem 2.2 [1; see also 18]. An orthoalgebra satisfying the Riesz property is a Boolean algebra.

Proof. Let $\mathcal{A}$ be an orthoalgebra satisfying the Riesz property. First we prove that $\forall a, b \in A$: if $a \perp b$, then $a \vee b$ exists and is equal to $a \oplus b$. It then easily follows that $\mathcal{A}$ is an orthomodular poset.

Let $a \perp b$. Clearly $a, b \leq a \oplus b$. Let $a, b \leq x$. By Theorem 2.1, there is a Mackey decomposition a_1, b_1, c of a, b such that $a_1 \oplus b_1 \oplus c \leq x$. But $a \perp b$ implies $c \perp c$, hence $c = 0$, so that $a_1 = a$, $b_1 = b$. This implies $a \oplus b \leq x$ whenever $a, b \leq x$, i.e., $a \oplus b = a \vee b$. Therefore $\mathcal{A}$ is an orthomodular poset. Since any two elements are compatible, $\mathcal{A}$ is a Boolean algebra.

2.1. Lattice-Ordered Effect Algebras

Let us now consider effect algebras that are *lattice-ordered* (in other words, the partial order $\leq$ gives rise to a lattice) (for basic properties see, e.g., refs. 1, 11, and 8).

Needless to recall that $E(\mathcal{H})$ is not lattice-ordered.

Lemma 2.6. The properties of being lattice-ordered and the Riesz property are incomparable.

An example of a lattice ordered effect algebra which does not satisfy the Riesz property is the “diamond” [8]. (Recall that the diamond is an effect algebra consisting of elements $\{0, 1, a, b\}$, where $a \perp a$, $b \perp b$, $a \oplus a = b \oplus b = 1$, $x \oplus 0 = x$, $x \in \{0, 1, a, b\}$.) An example of an affect algebra satisfying the Riesz property which is not a lattice is an interval $[0, a]$, where *a* is a positive element in a Riesz group which is not a lattice (see ref. 7 for such examples).

Theorem 2.3. For a lattice-ordered effect algebra the following conditions are equivalent:

1. Riesz.
2. Mackey.
3. $a \wedge b = \mathbf{0}$ implies $a \perp b$.

For the proof, see refs. 1 and 18.

In any lattice-ordered effect algebra one can define a total operation $\boxplus$:

$$a \boxplus b := a \oplus (a' \wedge b)$$

Notice that our operation $\boxplus$ is well defined because $a \perp (a' \wedge b)$.

Theorem 2.4. In any lattice-ordered effect algebra

$$a \leq b \qquad \text{iff} \quad a' \boxplus b = 1$$

As a consequence, lattice-ordered effect algebras have a "good" polynomial conditional.

Lemma 2.7. A lattice-ordered effect algebra satisfies the Riesz property iff the operation $\boxplus$ is commutative.

Definition 2.7. An effect algebra $\mathcal{A} = \langle A, \oplus, \mathbf{1}, 0\rangle$ is an *MV-effect algebra* iff there is an MV-algebra $\mathcal{M} = \langle M, \boxplus_M, {}'^M, \mathbf{1}_M, \mathbf{0}_M\rangle$ that represents a "reorganization" of $\mathcal{A}$. In other words:

1. $M = A$.
2. $a^{'M} = a'$.
3. $\mathbf{1}_M = \mathbf{1}$; $\mathbf{0}_M = 0$.
4. $\exists(a \oplus b)$ implies $a \boxplus_M b = a \oplus b$.
5. $a \leq_M b$ implies $a \leq b$.

For the proof of the next theorem, see ref. 18 (compare also with refs. 2 and 1).

Theorem 2.5. An effect algebra $\mathcal{A}$ is an MV-effect algebra iff $\mathcal{A}$ is lattice ordered and satisfies the Mackey property.

As is well known, an orthomodular poset is a Boolean algebra if and only if it satisfies the Mackey property. Theorem 2.5 shows that a similar result holds for a lattice-ordered effect algebra. It turns out that a much stronger notion of compatibility is needed to make a general effect algebra an MV-algebra.

Theorem 2.6. An effect algebra $\mathcal{A}$ is an MV-effect algebra iff $\forall a, b$ there is a Mackey decomposition a_1, b_1, c such that $a, b \le x$ implies $a_1 \oplus b_1 \oplus c \le x$.

Proof. It is easy to see that $a_1 \oplus b_1 \oplus c = a \vee b$, so that $\mathcal{A}$ is a lattice. Theorem 2.5 implies that $\mathcal{A}$ is an MV-effect algebra.

Conversely, in any MV algebra, $a \wedge b$, $(a \boxplus b')'$, $(b \boxplus a')'$ is a Mackey decomposition of a, b and $(a \wedge b) \boxplus (a \boxplus b')' \boxplus (b \boxplus a')' = a \vee b$. ■

3. A CHARACTERIZATION OF THE RANGE OF AN OBSERVABLE

In this section, we will discuss the following question: under what conditions can a subset F of elements of an effect algebra $\mathcal{A} = (A, \oplus, \mathbf{1}, \mathbf{0})$ be embedded into the range of a (finitely additive) observable? We will answer the question by using projective limits of simple observables (equivalently, partitions of unity in A) in a similar way as in ref. 13.

Let $(\Omega, \mathcal{B}(\Omega))$ be a nonempty set Ω and an algebra $\mathcal{B}(\Omega)$ of subsets of Ω. We will consider $(\mathcal{B}(\Omega), \mathcal{A})$-observables, i.e., morphisms from $\mathcal{B}(\Omega)$ to A according to Definition 1.3.

Definition 3.1. An observable $\alpha:\mathcal{B}(\Omega) \to A$ is *simple* iff there is a finite subset $\Omega_0 = \{\omega_1, \ldots, \omega_n\}$ of Ω such that each singleton $\{\omega_i\} \in \mathcal{B}(\Omega)$, for $i = 1,2, \ldots, n$, and $\alpha(\omega_1) \oplus \cdots \oplus \alpha(\omega_n) = \mathbf{1}$.

A *finite partition of unity* in A is a finite sequence $a_1, \ldots, a_n$ of elements of A such that $\oplus_{i \le n} a_i = \mathbf{1}$ (i.e., the latter sum exists end equals $\mathbf{1}$).

Lemma 3.1. There is one-to-one correspondence between simple observables and finite partitions of unity in A.

Proof. If α: $\mathcal{B}(\Omega) \to$ A is a simple observable, there is a finite set $\Omega_0 :=$ $\{\omega_1, \ldots, \omega_n\}$ of points in Ω such that each singleton $\{\omega_i\}$, for $i = 1, 2, \ldots,$ n, belongs to $\mathcal{B}(\Omega)$ and $R := \{\alpha(\omega_1), \ldots, \alpha(\omega_n)\}$ is a partition of unity. Let us notice that Ω_0 and any of its subsets are elements of $\mathcal{B}(\Omega)$, i.e., the whole power set 2^{Ω_0} of Ω_0 is contained in $\mathcal{B}(\Omega)$. Therefore, we can introduce α^R: $2^{\Omega_0} \to$ A defined as $\alpha^R(X) = \oplus_{\{i:\omega_i \in X\}} \alpha(\omega_i)$, $X \in 2^{\Omega_0}$. Then α^R is an observable which coincides with α on 2^{Ω_0}, this latter being such that $\alpha(\Omega_0) = \mathbb{1}$ [whence $\alpha(\Omega \backslash \Omega_0) = \mathbb{0}$], i.e., the *support* of α is Ω_0. For these reasons we can identify the original A-valued observable α on $\mathcal{B}(\Omega)$ with the A-valued observable α^R on 2^{Ω_0}.

Conversely, let $R := \{a_1, \ldots, a_n\}$ be a finite partition of unity in A. Let $\Omega_0 = \{\omega_1, \ldots, \omega_n\}$ be a finite set equipped with the algebra of subsets 2^{Ω_0}, and define, $\forall X \in 2^{\Omega_0}$, $\alpha(X) = \oplus_{\{i:\omega_i \in X\}} a_i$. It is easy to check that α: $2^{\Omega_0} \to A$ is a simple A-valued observable on 2^{Ω_0}. ■

Definition 3.2. (a) Let $(D, \leq)$ be a directed set and Ω_i a finite set for each $i \in D$. Whenever $i, j \in D$ and $i \leq j$, let there be given a mapping $g_{i,j}$: $\Omega_j \to \Omega_i$ and denote by $\mathcal{G} := \{g_{i,j}: i, j \in D \text{ and } i \leq j\}$ the collection of all such mappings. The pair $((\Omega_i)_{i\in D}, \mathcal{G})$ is called a *projective system of finite sets* if the following conditions hold:

(i) $g_{i,i}$ is the identity map on Ω_i for each $i \in D$.

(ii) $g_{i,j} \circ g_{j,k} = g_{i,k}$ whenever $i \leq j \leq k$.

(b) We say that $((\Omega_i, \alpha_i)_{i\in D}, \mathcal{G})$ is a *projective system of simple observables* if $((\Omega_i)_{i\in D}, \mathcal{G})$ is a projective system of finite sets and for each $i \in D$, α_i: $2^{\Omega_i} \to A$ is a simple observable such that the following *compatibility condition* holds:

(iii) $\forall X \in 2^{\Omega_i}$, $\alpha_i(X) = \alpha_j(g_{i,j}^{-1}(X))$ whenever $i \leq j$.

Let $((\Omega_i)_{i\in D}, (g_{i,j})_{i\in D, i\leq j})$ be a projective system of finite sets with directed set D. If $\Omega_D := \times_{i\in D}\Omega_i$ is the Cartesian product of the Ω_i, let Ω be its subset consisting of those elements, called *threads*, $\Omega = \{(\omega_i)_{i\in D} \in \Omega_D$ such that $\forall i, j \in D$ satisfying $i \leq j$ we have $g_{i,j}(\omega_j) = \omega_i\}$. Then Ω is called the *projective limit of the system* $((\Omega)_i)_{i\in D}, (g_{i,j})_{i\in D, i\leq j})$ and denote $\Omega =: \lim_{\leftarrow}(\Omega_i, g_{i,j})$ [19]. For this space we consider the *i-projection* g_i: $\Omega \to \Omega_i$, $\omega \mapsto \omega_i$, $\forall i \in D$. Trivially, $g_i = g_{i,j} \circ g_j$ whenever $i \leq j$.

In general, Ω can be very small (or empty) even if each $g_{i,j}$ is an onto mapping. In order to overcome this difficulty, the following (sufficient) condition (due to Bochner) is useful.

Definition 3.3. A projective system of finite sets $((\Omega_i)_{i\in D}, (g_{i,j})_{i\in D, i\leq j})$, is said to satisfy the *sequential maximality condition* (s.m. condition) if for each sequence $i_1 \leq i_2 \leq \cdots$ in D and any point $\omega = (\omega_i)_{i\in D}, \in \Omega_D$ such that

$$g_{i_n, i_{n+1}}(\omega_{i_{n+1}}) = \omega_{i_n}, \qquad n \geq 1$$

we have $\omega \in \Omega$.

It is not hard to verify that the s.m. condition holds if each Ω_i is a nonempty compact Hausdorff space (in particular, if it is finite) and all $g_{i,j}$ are onto mappings [19]. In the latter case $\emptyset \neq \Omega \subset \Omega_D$ is also compact.

Theorem 3.1. Let $\mathcal{A} = \langle A, \oplus, \mathbf{1}, \mathbf{0} \rangle$ be an effect algebra. A subset K of A can be embedded into a range of an observable α if and only if there is a projective system $((\Omega_i, \alpha_i)_{i\in D}, \mathcal{G})$ of simple observables such that for each $k \in K$ there is an $i \in D$ such that $k \in Range(\alpha_i)$.

Proof. (i) Assume that $K \subseteq Range(\alpha)$ for an observable α: $\mathcal{B}(\Omega) \to A$, where $\mathcal{B}(\Omega)$ is a Boolean algebra of subsets of a nonempty set Ω [i.e., for every $k \in K$, there exists $X_k \in \mathcal{B}(\Omega)$ such that $\alpha(X_k) = k$]. Let $\mathcal{R}$ be the set of all finite disjoint partitions of Ω with elements in $\mathcal{B}(\Omega)$ [in particular, for every $k \in K$ the pair $R_k := \{X_k, \Omega\backslash X_k\}$ is a finite partition of Ω, i.e., an element of $\mathcal{R}$]. For each $R = \{X_i\}_{i \le n} \in \mathcal{R}$, the mapping $R \ni X_i \mapsto \alpha(X_i) \in A$ defines uniquely a simple observable α^R: $2^R \to A$ by the law $\forall \{X_1, X_2, \ldots, X_j\} \subseteq R$, $\alpha^R(\{X_1, X_2, \ldots, X_j\}) = \alpha^R(X_1) \oplus \alpha^R(X_2) \oplus \ldots \oplus \alpha^R(X_j)$ [in particular, $\alpha^{R_k}(X_k) = k$]. Let $R_1, R_2 \in \mathcal{R}$ and assume that R_1 is a refinement of R_2, that is, for every $X_i \in R_1$ there is (exactly one) $Y_j \in R_2$, such that $X_i \subseteq Y_j$. We then write $R_2 \le R_1$. For $R_2 \le R_1$ the function $g_{R_1R_2}$: $R_1 \to R_2$, $X_i \mapsto Y_j$ (with $X_i \subseteq Y_j$) is then a uniquely defined onto mapping. If $R_2 \le R_1$ and $R_3 \le R_2$, then also $R_3 \le R_1$ and $g_{R_3R_1} = g_{R_3R_2} \circ g_{R_2R_1}$. Finally, for any two partitions $R_1, R_2 \in \mathcal{R}$ the partition $R = \{X_i \cap Y_j: X_i \in R_1, Y_j \in R_2\}$ belongs to $\mathcal{R}$, and is a common refinement of R_1 and R_2, that is, $R_1 \le R$, $R_2 \le R$, and so $(\mathcal{R}, \le)$ is a directed set. Let $\mathcal{G} := \{g_{SR}: R, S \in \mathcal{R}, R \le S\}$. It is now immediate to check that $((R, \alpha^R)_{R \in \mathcal{R}}, \mathcal{G})$ is a projective system of simple observables. By construction, $K \subseteq \cup_{R \in \mathcal{R}}$ *Range*(α^R).

Conversely, let $((\Omega_i, \alpha_i)_{i \in D}, \mathcal{G})$ be a projective system of simple observables. Then, the Bochner s.m. condition is satisfied and, by Proposition 4 in ref. 19, p. 120, the projective limit $\Omega = \lim_{i \leftarrow}$ is a compact Hausdorff space. Let g_i: $\Omega \to \Omega_i$ be the natural coordinate projection, $i \in D$. The s.m. condition implies that $g_i(\Omega) = \Omega_i$. Define $\mathcal{B}(\Omega)_0 := \cup_i g_i^{-1}(\mathcal{B}(\Omega)_i)$. Then $\mathcal{B}(\Omega)_0$ is an algebra of subsets of Ω, which is generated by the cylinder sets of Ω with bases in $\mathcal{B}(\Omega)_j$, $i \in D$, where $\mathcal{B}(\Omega)_i := 2^{\Omega_i}$. Let $X \in \mathcal{B}(\Omega)_0$. If $X \in g_i^{-1}(\mathcal{B}(\Omega)_i) \cap g_j^{-1}(\mathcal{B}(\Omega)_j)$, then there exist $Y_1 \in \mathcal{B}(\Omega)_i$, $Y_2 \in \mathcal{B}(\Omega)_j$ such that $X = g_i^{-1}(Y_1) = g_j^{-1}(Y_2)$. Since D is directed, there exists an $l \in D$ such that $l \ge i$, $l \ge j$, and by compatibility of the mappings, we have $g_i = g_{il} \circ g_l$, $g_j = g_{jl} \circ g_l$. Hence

$$\begin{aligned} g_l^{-1}(g_{il}^{-1}(Y_1)) &= g_i^{-1}(Y_1) = X \\ &= g_j^{-1}(Y_2) = g_l^{-1}(g_{jl}^{-1}(Y_2)) \end{aligned}$$

Since g_l: $\Omega \to \Omega_l$ is onto, it follows that g_l^{-1}: $\mathcal{B}(\Omega)_l \to \mathcal{B}(\Omega)_0$ is one-to-one, so that the latter equations imply that

$$g_{il}^{-1}(Y_1) = g_{jl}^{-1}(Y_2)$$

Thus

$$\begin{aligned} \alpha_i(Y_1) &= \alpha_l(g_{il}^{-1}(Y_1)) \\ &= \alpha_l(g_{jl}^{-1}(Y_2)) = \alpha_j(Y_2) \end{aligned}$$

by the compatibility of observables. If we set $\alpha(X) := \alpha_i(Y_1) = \alpha_j(Y_2)$, $X \in \mathcal{B}(\Omega)_0$, then α is unambiguously defined on $\mathcal{B}(\Omega)_0$. Clearly, $\alpha(\Omega) = \alpha_i(\Omega_i) = 1$, and α is finitely additive on $\mathcal{B}(\Omega)_0$. Moreover, the prescription also implies that $\alpha_i = \alpha \circ g_i^{-1}$, $i \in D$. Hence $\forall i \in D$, *Range*$(\alpha_i) \subset$ *Range*(α). ∎

ACKNOWLEDGMENT

This work has been done during the stay of the fourth author, S.P., at the Dipartimento di Filosofia, Università di Firenze, supported by a grant of CNR. The author expresses her thanks.

The work has been partially supported by grant VEGA 2/4033/97.

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